二元序列集的非周期完全平方模糊函数理论界和最优构造

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摘 要：扩频序列集是直接-序列码分多址系统中的关键组成部分，其性能可通过完全平方相关进行评估。在高速移动场景中，信号在传输过程中会产生多普勒效应，需同时考虑序列的时移和多普勒移位。此时，应使用二维模糊函数替代一维相关函数。该文主要研究了二元序列集的非周期完全平方模糊函数(Aperiodic Total Squared Ambiguity Function, ATSAF)，推导了二元序列集的ATSAF理论下界。基于Hadamard矩阵、非周期互补集和特殊序列，设计了几类达到ATSAF理论下界的最优二元序列集。

关键词：非周期完全平方模糊函数；模糊函数；完全平方相关

中图分类号：TN911.2 **文献标识码**：A **文章编号**：1009-5896(2026)12-0001-09

DOI: 10.11999/JEIT251327 **CSTR**: 32379.14.JEIT251327

1 引言

随着多输入多输出、码分多址、正交频分复用、非正交多址和通信感知一体化等技术的不断发展，学术界和工业界对设计具有良好相关性的序列集产生了浓厚的研究兴趣。在活跃设备检测中，具有最优相关性的序列集通常用于设计压缩感知矩阵[1]。此外，具有良好相关性的互补码可以实现通信雷达的无扰传输[2]。最大相关幅度和完全平方相关[3] (Total Squared Correlation, TSC)是序列设计中常用的两个评价指标。其中，最大相关幅度关注序列集相关性的最坏情况，而TSC关注序列集的整体相关性。本文主要关注后者。

对于由K条长度为L的二元序列组成的序列集$\boldsymbol{S}$，为了减少通信/雷达系统的整体干扰水平，必须尽可能减小TSC($\boldsymbol{S}$)。1974年，Welch[4]提出了序列集的(非)周期TSC理论下界。在过载系统中，当序列集满足$\mathrm{TSC}(\boldsymbol{S})=K^2/L$时，该序列集被称为Welch界等式(Welch Bound Equality, WBE)集。关于WBE序列集的设计算法和研究可参考文献[5–7]。继Welch的工作后，陆续有学者关注序列集的TSC理论界。2003年，Karystinos等人[8]分别在轻载系统和过载系统下改进了Welch界，即对于任意的K和L，提出了更紧的二元序列集的TSC理论界。通过Hadamard变换，还设计了几类满足TSC理论界的最优二元序列集。然而，对于$L=K\equiv 1\ (\mathrm{mod}4)$，$L=K\equiv 2(\mathrm{mod}4)$，$L+1=K\equiv 2(\mathrm{mod}4)$和$L=K+1\equiv 2(\mathrm{mod}4)$这4种情况，未能设计出达到理论下界的最优二元序列集。2003年，丁存生等人[9]通过Hadamard变换，设计出了满足Karystinos等人提出的TSC理论下界的最优二元序列集，解决了除$L=K\equiv 1(\mathrm{mod}4)$情况外的最优二元序列集设计问题。2009年，Li等人[10]提出了四元序列集的TSC理论下界，并设计了达到该理论界的最优四元序列集。2021年，考虑到两小区场景，由于小区内与小区间信道强度的差异，Wang等人[11]通过引入小区间干扰因子β作为权重，对序列间的加权相关平方和进行分析，提出了拓展完全平方相关(Extended Total Squared total Correlation, ETSC)理论下界，也称为拓展Welch界，其表征了系统中的总干扰。此外，文献[11]设计了能够满足ETSC理论下界的序列集。然而，ETSC仅适用于两小区场景。对于多小区场景，Shen等人[12]提出了广义的拓展完全平方相关(Generalized Extended Total Squared Correlation, GETSC)理论下界，也称为广义的拓展Welch界。此外，文献[12]也设计了能够满足GETSC理论下界的序列集。无论是基于最小二乘估计还是最小均方误差估计，达到GETSC理论下界的序列集比WBE集能实现更精确的信道估计[12]。

TSC研究序列集的内积相关性，适用于同步通信系统。当考虑异步通信时，序列集在非零时延的

收稿日期：2025-12-16；改回日期：2026-02-05；网络出版：2026-03-01
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基金项目：国家自然科学基金项目(12401695)，四川省自然科学基金项目(2026NSFSC0776, 2025ZNSFSC0872)，四川省中央引导地方项目(2025ZYD0151)，四川省科技计划项目(2024NSFTD0015)
Foundation Items: The National Natural Science Foundation of China (12401695), Sichuan Province Natural Science Foundation Project (2026NSFSC0776, 2025ZNSFSC0872), Central Government Guides Local Science and Technology Development Project in Sichuan Province (2025ZYD0151), Sichuan Provincial Science and Technology Program Project (2024NSFTD0015)

相关函数值也对系统性能起着重要作用，仅考虑内积的TSC不再适用。对于周期和非周期相关函数，Ganapathy等人[13,14]引入周期完全平方相关(Periodic Total Squared Correlation, PTSC)和非周期完全平方相关(Aperiodic Total Squared Correlation, ATSC)，将序列集中所有相关函数值的平方总和作为度量，分别推导了PTSC和ATSC的理论下界，且设计了多种达到理论下界的最优二元序列集。对于奇周期相关函数，Liu等人[1]引入奇完全平方相关(Odd Total Squared Correlation, OTSC)的概念，推导了OTSC的理论下界，并设计了多种达到理论下界的最优二元序列集。在大规模设备活跃检测中[1]，达到PTSC或OTSC理论下界的序列集比高斯随机矩阵可支持的用户数更多，且具有更小的虚警概率和漏检概率。

在现代高速移动通信和卫星通信中，多普勒效应显著，导致接收信号产生时间和多普勒移位，进而接收信号严重失真。为了抵抗多普勒，学术界近年来越来越关注抗多普勒序列设计，以应对各种移动信道中的多普勒效应[15]。文献[16]和文献[17]基于内积定理和恒模序列的模糊函数零时延性质，推导了周期/非周期/奇周期最大模糊幅度的理论下界。文献[16–20]构造了能够满足或渐近满足模糊函数理论下界的序列集。此外，受互补序列启发，文献[21]考虑到互补序列集中的多普勒移位，提出了抗多普勒互补序列(Doppler Resilient Complementary Sequence, DRCS)，推导了周期/非周期/奇周期DRCS最大模糊幅度的理论下界，且通过正交矩阵、循环佛罗伦萨矩形、差集构造了几类渐近最优的周期/非周期/奇周期DRCS。

当考虑多普勒频移时，已有工作聚焦于序列集的最大模糊幅度，尚不涉及完全平方模糊函数(Total Squared Ambiguity Function, TSAF)。受此启发，本文研究二元序列集的ATSAF，通过序列集的非周期时相循环扩展矩阵推导了ATSAF的理论下界，并设计了几类达到ATSAF理论下界的最优二元序列集。

2 基本概念

符号说明：$\xi_L=\exp(2\pi\sqrt{(-1)}/L)$表示$L$次本原单位根；$\mathbb{Z}_L=\{0,1,\cdots,L-1\}$表示模$L$剩余类环；$x^*$表示复数$x$的共轭；$\boldsymbol{x}^{\mathrm{T}}$和$\boldsymbol{x}^{\mathrm{H}}$分别表示向量$\boldsymbol{x}$的转置和共轭转置；$\boldsymbol{x}\odot\boldsymbol{y}$ 表示$\boldsymbol{x}$与$\boldsymbol{y}$的Hadamard积；长度为L的序列$\boldsymbol{s}=[s(0),\,s(1),\cdots,s(L-1)]^{\mathrm{T}}$表示为$L\times 1$的列向量；$\langle x\rangle_N$表示整数$x$模整数$N$的余数；序列$\boldsymbol{s}$的$v$位多普勒偏移表示为$\boldsymbol{s}^{(v)}=\boldsymbol{s}\odot[\xi_{\mathrm{L}}^{0\times v},\xi_L^{1\times v},\cdots,\xi_L^{(L-1)\times v}]^{\mathrm{T}}$，其中，$0\le v<V\le L$，$V$表示最大多普勒移位；序列$\boldsymbol{s}$的$l$位非周期循环时移表示为

$$T(\boldsymbol{s},l)=[s(l),s(l+1),\cdots,s(L-1),\underbrace{0,\cdots,0}_{L-1},s(0),s(1),\cdots s(l-1)]^{\mathrm{T}}$$

其中，$0\le l\le 2L-2$。

定义1[16] 设$\boldsymbol{s}=[s(0),s(1),\cdots,s(L-1)]^{\mathrm{T}}$和$\boldsymbol{t}=[t(0),t(1),\cdots,t(L-1)]^{\mathrm{T}}$是两条长度为$L$的二元序列，它们在时延$\tau$和多普勒偏移$v$处的非周期互模糊函数定义为

$$\mathrm{AF}_{\boldsymbol{s},\boldsymbol{t}}(\tau,v)=\begin{cases}\displaystyle\sum_{i=0}^{L-1-\tau}s(i)t^*(i+\tau)\xi_L^{vi}, & 0\le\tau\le L-1\\ \displaystyle\sum_{i=-\tau}^{L-1}s(i)t^*(i+\tau)\xi_L^{vi}, & 1-L\le\tau<0\\ 0, & |\tau|\ge L\end{cases}\tag{1}$$

其中，$0\le v\le L-1$。当$\boldsymbol{s}=\boldsymbol{t}$时，称$\mathrm{AF}_{\boldsymbol{s},\boldsymbol{t}}(\tau,v)$为非周期自模糊函数。若$v=0$，则$\mathrm{AF}_{\boldsymbol{s},\boldsymbol{t}}(\tau,0)$表示非周期互相关函数，记为$C_{\boldsymbol{s},\boldsymbol{t}}(\tau)$。当$\boldsymbol{s}=\boldsymbol{t}$时，称$C_{\boldsymbol{s},\boldsymbol{t}}(\tau)$为非周期自相关函数，记为$C_{\boldsymbol{s}}(\tau)$。

定义2[3] 对于K条长度为L的二元序列集$\boldsymbol{S}=[\boldsymbol{s}_1,\boldsymbol{s}_2,\cdots,\boldsymbol{s}_K]$，其TSC定义为

$$\mathrm{TSC}(\boldsymbol{S})=\sum_{i=1}^{K}\sum_{j=1}^{K}\left|\boldsymbol{s}_i^{\mathrm{H}}\boldsymbol{s}_j\right|^2\tag{2}$$

定义3 设$V\le L$是一个正整数。对于$0\le l\le 2L-2$，定义序列集$\boldsymbol{S}$的l位非周期循环移位为

$$T(\boldsymbol{S},l)=[T(\boldsymbol{s}_1,l),T(\boldsymbol{s}_2,l),\cdots,T(\boldsymbol{s}_K,l)]\tag{3}$$

对于$0\le v\le V-1$，令

$$\boldsymbol{F}_L^{(v)}=\begin{bmatrix}1 & 1 & \cdots & 1\\ \xi_{\mathrm{L}}^{v} & \xi_{\mathrm{L}}^{v} & \cdots & \xi_L^{v}\\ \vdots & \vdots & \ddots & \vdots\\ \xi_L^{v(L-1)} & \xi_L^{v(L-1)} & \cdots & \xi_L^{v(L-1)}\end{bmatrix}_{L\times K}\tag{4}$$

进一步地，$\boldsymbol{S}$的非周期时相循环扩展矩阵定义为$\boldsymbol{S}_a=[\varPhi_0,\varPhi_1,\cdots,\varPhi_{V-1}]$，其中

$$\begin{aligned}\varPhi_v&=[\boldsymbol{U}^{(0,v)},\boldsymbol{U}^{(1,v)},\cdots,\boldsymbol{U}^{(L-1,v)}]\\ \boldsymbol{U}^{(l,v)}&=T(\boldsymbol{S},l)\odot T(\boldsymbol{F}_{\mathrm{L}}^{(v)},l)\end{aligned}\tag{5}$$

最后，$\boldsymbol{S}$的ATSAF定义为

$$\mathrm{ATSAF}(\boldsymbol{S})=\mathbf{TSC}(\boldsymbol{S}_a)\tag{6}$$

注1 根据定义1和文献[16]可知，序列集$\boldsymbol{S}=[\boldsymbol{s}_1,\boldsymbol{s}_2,\cdots,\boldsymbol{s}_K]$的ATSAF可以等价地表示为

$$\begin{aligned}\mathrm{ATSAF}(\boldsymbol{S}) &= \left\|\boldsymbol{S}_a\boldsymbol{S}_a^{\mathrm{H}}\right\|_{\mathrm{F}}^2 \\ &= \sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{\tau=1-L}^{L-1}\sum_{v=1-V}^{V-1}(L-|\tau|) \\ &\quad (V-|v|)\left|\mathrm{AF}_{\boldsymbol{s}_i,\boldsymbol{s}_j}(\tau,v)\right|^2 \end{aligned} \tag{7}$$

其中，$1\le V\le L$。

定义4[22]　设$\boldsymbol{S}=[\boldsymbol{s}_1,\boldsymbol{s}_2,\cdots,\boldsymbol{s}_K]$是含有$K$条长度为$L$的序列集，若

$$\sum_{i=1}^{K}C_{\boldsymbol{s}_i}(\tau)=0,\tau\ne 0 \tag{8}$$

则称$\boldsymbol{S}$为非周期互补集(Aperiodic Complementary Set, ACS)。

由文献[22]可知：(1) 二元ACS存在的必要条件是$KL\equiv 0(\mathrm{mod}4)$且$K\equiv 0(\mathrm{mod}2)$；(2) ACS是ATSC最优序列集[14]，即可以达到ATSC理论下界。文献[23]和文献[24]提供了大量ACS的直接或递归构造。

3　非周期完全平方模糊函数理论界

对于含有K条长度为L的二元序列集$\boldsymbol{S}=[\boldsymbol{s}_1,\boldsymbol{s}_2,\cdots,\boldsymbol{s}_K]$，根据ATSAF的定义及矩阵**TSC**度量的“行列等价性[25]”可知，ATSAF($\boldsymbol{S}$)的等价形式为

$$\begin{aligned}\mathrm{ATSAF}(\boldsymbol{S}) &= \left\|\boldsymbol{S}_a\boldsymbol{S}_a^{\mathrm{H}}\right\|_{\mathrm{F}}^2=\left\|\boldsymbol{S}_a^{\mathrm{H}}\boldsymbol{S}_a\right\|_{\mathrm{F}}^2 \\ &= \sum_{i=1}^{2L-1}\sum_{j=1}^{2L-1}\left|\boldsymbol{d}_i\boldsymbol{d}_j^{\mathrm{H}}\right|^2 \\ &= K^2L^2V^2(2L-1)+\sum_{i=1}^{2L-1}\sum_{\substack{j=1\\ j\ne i}}^{2L-1}\left|\boldsymbol{d}_i\boldsymbol{d}_j^{\mathrm{H}}\right|^2 \end{aligned} \tag{9}$$

其中，$\boldsymbol{d}_i$表示$\boldsymbol{S}_a^{\mathrm{T}}$的第$i$行。

通过分析式(9)可知，只需计算所有不同K和L情况下$\left|\boldsymbol{d}_i\boldsymbol{d}_j^{\mathrm{E}}\right|$的下界，即可确定序列集$\boldsymbol{S}$的ATSAF下界。

引理1　对于含有K条长度为L的二元序列集$\boldsymbol{S}=[\boldsymbol{s}_1,\boldsymbol{s}_2,\cdots,\boldsymbol{s}_K]$，且$V\le L$，则

$$\begin{aligned}&\mathrm{ATSAF}(\boldsymbol{S}) \\ &= K^2L^2V^2(2L-1)+(2L-1)\sum_{j=2}^{2L-1}\left|\boldsymbol{d}_1\boldsymbol{d}_j^{\mathrm{H}}\right|^2 \\ &= K^2L^2V^2(2L-1)+2(2L-1)\sum_{j=2}^{L}\left|\boldsymbol{d}_1\boldsymbol{d}_j^{\mathrm{H}}\right|^2 \end{aligned} \tag{10}$$

证明　对于$1\le i\ne j\le 2L-1$，根据$\boldsymbol{S}_a$的定义，有

$$\boldsymbol{d}_i\boldsymbol{d}_j^{\mathrm{H}}=\begin{cases}\boldsymbol{d}_1\boldsymbol{d}_{2L-(i-j)}^{\mathrm{H}}=\displaystyle\sum_{v=0}^{V-1}\sum_{k=1}^{K}\left(\boldsymbol{u}_k^{(0,v)}\right)^{\mathrm{T}} \\ \quad\cdot\left(\boldsymbol{u}_k^{(2L-1-(i-j),v)}\right)^*,i>j \\ \boldsymbol{d}_1\boldsymbol{d}_{j-i+1}^{\mathrm{H}}=\displaystyle\sum_{v=0}^{V-1}\sum_{k=1}^{K}\left(\boldsymbol{u}_k^{(0,v)}\right)^{\mathrm{T}}\left(\boldsymbol{u}_k^{(j-i,v)}\right)^*, \\ \quad\cdot i<j\end{cases} \tag{11}$$

其中，$\boldsymbol{u}_k^{(l,v)}$表示$\boldsymbol{U}^{(l,v)}$的第k列。因此，

$$\left|\boldsymbol{d}_1\boldsymbol{d}_j^{\mathrm{H}}\right|=\left|\boldsymbol{d}_1\boldsymbol{d}_{2L-j+1}^{\mathrm{H}}\right|,L+1\le j\le 2L-1 \tag{12}$$

将式(11)和式(12)代入式(9)，可得

$$\begin{aligned}\sum_{i=1}^{2L-1}\sum_{\substack{j=1\\ j\ne i}}^{2L-1}\left|\boldsymbol{d}_i\boldsymbol{d}_j^{\mathrm{H}}\right|^2 &= (2L-1)\sum_{j=2}^{2L-1}\left|\boldsymbol{d}_1\boldsymbol{d}_j^{\mathrm{H}}\right|^2 \\ &= 2(2L-1)\sum_{j=2}^{L}\left|\boldsymbol{d}_1\boldsymbol{d}_j^{\mathrm{H}}\right|^2 \end{aligned} \tag{13}$$

因此，引理1得证。　　证毕

由引理1可知，只需计算不同K和L情况下$\left|\boldsymbol{d}_1\boldsymbol{d}_j^{\mathrm{E}}\right|$的下界，即可确定序列集$\boldsymbol{S}$的ATSAF下界。

定理1　设$1\le V\le L$。对于含有K条长度为L的二元序列集$\boldsymbol{S}=[\boldsymbol{s}_1,\boldsymbol{s}_2,\cdots,\boldsymbol{s}_K]$，其 ATSAF理论下界为

$$\mathrm{ATSAF}(\boldsymbol{S})\ge\begin{cases}K^2L^2V^2(2L-1), & KL\equiv 0(\mathrm{mod}4),K\equiv 0(\mathrm{mod}2) \\ K^2L^2V^2(2L-1)+(2L-1)(L-V)V, & K\equiv 1(\mathrm{mod}2),L\equiv 1(\mathrm{mod}2) \\ K^2L^2V^2(2L-1)+4(2L-1)(L-V)V, & K\equiv 2(\mathrm{mod}4),L\equiv 1(\mathrm{mod}2) \\ K^2L^2V^2(2L-1)+(2L-1)LV, & KL\equiv 0(\mathrm{mod}4),K\equiv 1(\mathrm{mod}2),1\le V\le L/2 \\ K^2L^2V^2(2L-1)+(2L-1)(L-V)L, & KL\equiv 0(\mathrm{mod}4),K\equiv 1(\mathrm{mod}2),L/2<V\le L \\ K^2L^2V^2(2L-1)+(2L-1)(3L-4V)V, & K\equiv 1(\mathrm{mod}2),L\equiv 2(\mathrm{mod}4),1\le V\le L/2 \\ K^2L^2V^2(2L-1)+(2L-1)(L-V)(4V-L), & K\equiv 1(\mathrm{mod}2),L\equiv 2(\mathrm{mod}4),L/2<V\le L\end{cases} \tag{14}$$

证明　当$V=L$时，根据$\boldsymbol{S}_a$的定义，容易得到

$$\left|\boldsymbol{d}_1\boldsymbol{d}_j^{\mathrm{H}}\right|=\left|\sum_{v=0}^{L-1}\sum_{k=1}^{K}\left(\boldsymbol{u}_k^{(0,v)}\right)^{\mathrm{T}}\left(\boldsymbol{u}_k^{(j-1,v)}\right)^*\right|=\left|\varphi(L,K,j)\sum_{v=0}^{L-1}\xi_L^{-(j-1)v}\right|=0,2\le j\le L \tag{15}$$

其中，

$$\varphi(L,K,j)=\left|\sum_{k=1}^{K}\left(\boldsymbol{u}_k^{(0,0)}\right)^{\mathrm{T}}\left(\boldsymbol{u}_k^{(j-1,0)}\right)^{*}\right| \tag{16}$$

当 $L\equiv 1(\mathrm{mod}2), 1\le V\le L$ 或$L\equiv 0(\mathrm{mod}2)$，$1\le V\le L/2$ 时，有

$$\left|\boldsymbol{d}_1\boldsymbol{d}_j^{\mathrm{H}}\right| =\begin{cases}\left|\varphi(L,K,j)\sum_{v=0}^{V-1}\xi_L^{-(j-1)v}\right|, 2\le j\le L\\ \left|\varphi(L,K,j)\sum_{v=0}^{V-1}\xi_L^{(L-j)v}\right|, L+1\le j\le 2L-1\end{cases} \tag{17}$$

当$L\equiv 0(\mathrm{mod}2), L/2<V\le L-1$时，有

$$\left|\boldsymbol{d}_1\boldsymbol{d}_j^{\mathrm{H}}\right| =\begin{cases}\left|\varphi(L,K,j)\sum_{v=V-(L/2)}^{(L/2)-1}\xi_L^{-(j-1)v}\right|,\\ \quad 2\le j\le L\\ \left|\varphi(L,K,j)\sum_{v=V-(L/2)}^{(L/2)-1}\xi_L^{(L-j)v}\right|,\\ \quad L+1\le j\le 2L-1\end{cases} \tag{18}$$

下面将K和L分为5种情况，以讨论不同条件下$\varphi(L,K,j)$的最小值，并推导了不同条件下序列集$\boldsymbol{S}$的ATSAF理论下界。

情况1：$KL\equiv 0(\mathrm{mod}4), K\equiv 0(\mathrm{mod}2)$。由文献[14]和式(12)可知

$$\varphi(L,K,j)\ge 0, 2\le j\le L \tag{19}$$

将式(19)代入式(17)和式(18)可知，$\left|\boldsymbol{d}_1\boldsymbol{d}_j^{\mathrm{H}}\right|\ge 0, 2\le j\le 2L-1$。因此，可得$\mathrm{ATSAF}(\boldsymbol{S})\ge K^2L^2V^2(2L-1)$。

情况2：$K\equiv 1(\mathrm{mod}2), L\equiv 1(\mathrm{mod}2)$。可得

$$\varphi(L,K,j)\ge\begin{cases}1, 2\le j\le L, j\equiv 1(\mathrm{mod}2)\\ 0, 2\le j\le L, j\equiv 0(\mathrm{mod}2)\\ 1, L+1\le j\le 2L-1, j\equiv 0(\mathrm{mod}2)\\ 0, L+1\le j\le 2L-1, j\equiv 1(\mathrm{mod}2)\end{cases} \tag{20}$$

将式(17)和式(20)代入式(10)，可得

$$\begin{aligned}&\mathrm{ATSAF}(\boldsymbol{S})\ge K^2L^2V^2(2L-1)+(2L-1)\\ &\left(\sum_{\substack{j=2\\ j\equiv 1(\mathrm{mod}2)}}^{L}\left|\sum_{v=0}^{V-1}\xi_L^{-(j-1)v}\right|^2+\sum_{\substack{j=L+1\\ j\equiv 0(\mathrm{mod}2)}}^{2L-1}\left|\sum_{v=0}^{V-1}\xi_L^{(L-j)v}\right|^2\right)\\ &=K^2L^2V^2(2L-1)+(2L-1)\sum_{j=1}^{L-1}\left|\sum_{v=0}^{V-1}\xi_L^{-jv}\right|^2\\ &=K^2L^2V^2(2L-1)+(2L-1)(L-V)V\end{aligned} \tag{21}$$

情况 3：$K\equiv 2(\mathrm{mod}4), L\equiv 1(\mathrm{mod}2)$。可得

$$\varphi(L,K,j)\ge\begin{cases}0\text{或}2, & 2\le j\le L\\ 0\text{或}2, & L+1\le j\le 2L-1\end{cases} \tag{22}$$

由文献[14]可知，当$2\le j\le(L+1)/2$时，$\varphi(L,K,j)$必须满足$\varphi(L,K,j)+\varphi(L,K,L-j+2)\ge 2$。因此，以下两种情况必然成立一种

$$\begin{aligned}&(\mathrm{I})\varphi(L,K,j)\ge 0, \text{且}\varphi(L,K,L-j+2)\ge 2\\ &(\mathrm{II})\varphi(L,K,j)\ge 2, \text{且}\varphi(L,K,L-j+2)\ge 0\end{aligned} \tag{23}$$

再根据式(12)的对称性可知，当$L+1\le j\le(3L-1)/2$时，$\varphi(L,K,j)$必须满足$\varphi(L,K,j)+\varphi(L,K,3L-j)\ge 2$。因此，以下两种情况必然成立一种

$$\begin{aligned}&(\mathrm{III})\varphi(L,K,j)\ge 0, \text{且}\varphi(L,K,3L-j)\ge 2\\ &(\mathrm{IV})\varphi(L,K,j)\ge 2, \text{且}\varphi(L,K,3L-j)\ge 0\end{aligned} \tag{24}$$

根据上述分析，由式(12)的对称性可知，$\varphi(L,K,j)$要么满足(I)和(IV)，要么满足(II)和(III)。不妨设$\varphi(L,K,j)$满足(I)和(IV)，将式(17)和式(22)代入式(10)，可得

$$\begin{aligned}&\mathrm{ATSAF}(\boldsymbol{S})\ge K^2L^2V^2(2L-1)+(2L-1)\\ &\left(\sum_{j=\frac{L+1}{2}+1}^{L}\left|2\sum_{v=0}^{V-1}\xi_L^{-(j-1)v}\right|^2+\sum_{j=L+1}^{\frac{3L-1}{2}}\left|2\sum_{v=0}^{V-1}\xi_L^{(L-j)v}\right|^2\right)\\ &=K^2L^2V^2(2L-1)+4(2L-1)\sum_{j=1}^{L-1}\left|\sum_{v=0}^{V-1}\xi_L^{-jv}\right|^2\\ &=K^2L^2V^2(2L-1)+4(2L-1)(L-V)V\end{aligned} \tag{25}$$

当$\varphi(L,K,j)$满足(II)和(III)时，证明同理。

情况 4：$KL\equiv 0(\mathrm{mod}4), K\equiv 1(\mathrm{mod}2)$。可得

$$\varphi(L,K,j)\ge\begin{cases}1, & 2\le j\le L, j\equiv 0(\mathrm{mod}2)\\ 0, & 2\le j\le L, j\equiv 1(\mathrm{mod}2)\\ 1, & L+1\le j\le 2L-1, j\equiv 1(\mathrm{mod}2)\\ 0, & L+1\le j\le 2L-1, j\equiv 0(\mathrm{mod}2)\end{cases} \tag{26}$$

当$1\le V\le L/2$时，将式(17)和式(26)代入式(10)，可得

$$\begin{aligned}&\mathrm{ATSAF}(\boldsymbol{S})\ge K^2L^2V^2(2L-1)+(2L-1)\\ &\left(\sum_{\substack{j=2\\ j\equiv 0(\mathrm{mod}2)}}^{L}\left|\sum_{v=0}^{V-1}\xi_L^{-(j-1)v}\right|^2+\sum_{\substack{j=L+1\\ j\equiv 1(\mathrm{mod}2)}}^{2L-1}\left|\sum_{v=0}^{V-1}\xi_L^{(L-j)v}\right|^2\right)\\ &=K^2L^2V^2(2L-1)+2(2L-1)\\ &\sum_{\substack{j=1\\ j\equiv 1(\mathrm{mod}2)}}^{L-1}\left(\sum_{\substack{v_1,v_2=0\\ v_1\ne v_2}}^{V-1}\xi_L^{j(v_2-v_1)}+\sum_{\substack{v_1,v_2=0\\ v_1=v_2}}^{V-1}\xi_L^{j(v_2-v_1)}\right)\\ &=K^2L^2V^2(2L-1)+(2L-1)LV\end{aligned} \tag{27}$$

当$L/2 < V \le L-1$时，将式(18)和式(26)代入式(10)，可得

$$\begin{aligned}\mathrm{ATSAF}(\boldsymbol{S}) &\ge K^2L^2V^2(2L-1)+(2L-1)\\ &\cdot\left(\sum_{\substack{j=2\\ j\equiv 0(\mathrm{mod}2)}}^{L}\left|\sum_{v=V-\frac{L}{2}}^{\frac{L}{2}-1}\xi_L^{-(j-1)v}\right|^2 + \sum_{\substack{j=L+1\\ j\equiv 1(\mathrm{mod}2)}}^{2L-1}\left|\sum_{v=V-\frac{L}{2}}^{\frac{L}{2}-1}\xi_L^{(L-j)v}\right|^2\right)\\ &= K^2L^2V^2(2L-1)+2(2L-1)\\ &\sum_{\substack{j=1\\ j\equiv 1(\mathrm{mod}2)}}^{L-1}\left(\sum_{\substack{v_1,v_2=V-\frac{L}{2}\\ v_1\ne v_2}}^{\frac{L}{2}-1}\xi_L^{j(v_2-v_1)}+\sum_{\substack{v_1,v_2=V-\frac{L}{2}\\ v_1=v_2}}^{\frac{L}{2}-1}\xi_L^{j(v_2-v_1)}\right)\\ &= K^2L^2V^2(2L-1)+(2L-1)(L-V)L \end{aligned}\tag{28}$$

当$V=L$时，容易得到，$\mathrm{ATSAF}(\boldsymbol{S})=K^2L^4(2L-1)$。因此，当$L/2<V\le L$时，有

$$\mathrm{ATSAF}(\boldsymbol{S})=K^2L^2V^2(2L-1)+(2L-1)(L-V)L$$

情况 5：$K\equiv 1(\mathrm{mod}2), L\equiv 2(\mathrm{mod}4)$。可得

$$\varphi(L,K,j) \ge \begin{cases}1, & 2\le j\le L, j\equiv 0(\mathrm{mod}2)\\ 1, & L+1\le j\le 2L-1, j\equiv 1(\mathrm{mod}2)\\ 0\text{或}2, & 2\le j\le L, j\equiv 1(\mathrm{mod}2)\\ 0\text{或}2, & L+1\le j\le 2L-1, j\equiv 0(\mathrm{mod}2)\end{cases}\tag{29}$$

由文献[14]可知，当$2\le j\le L/2, j\equiv 1(\mathrm{mod}2)$时，$\varphi(L,K,j)$必须满足$\varphi(L,K,j)+\varphi(L,K,L-j+2)\ge 2$。因此，以下两种情况必然成立一种

$$\begin{aligned}&(\mathrm{V})\varphi(L,K,j)\ge 0,\varphi(L,K,L-j+2)\ge 2\\ &(\mathrm{VI})\varphi(L,K,j)\ge 2,\varphi(L,K,L-j+2)\ge 0\end{aligned}\tag{30}$$

再根据式(12)的对称性可知，当$L+1\le j\le 3L/2, j\equiv 0(\mathrm{mod}2)$时，$\varphi(L,K,j)$必须满足

$$\varphi(L,K,j)+\varphi(L,K,3L-j)\ge 2$$

因此，以下两种情况必然成立一种

$$\begin{aligned}&(\mathrm{VII})\varphi(L,K,j)\ge 0,\varphi(L,K,3L-j)\ge 2\\ &(\mathrm{VIII})\varphi(L,K,j)\ge 2,\varphi(L,K,3L-j)\ge 0\end{aligned}\tag{31}$$

根据上述分析，由式(12)的对称性可知，$\varphi(L,K,j)$要么满足(V)和(VIII)，要么满足(VI)和(VII)。不妨设$\varphi(L,K,j)$满足(V)和(VIII)。当$1\le V\le L/2$时，将式(17)和式(29)代入式(10)，可得

$$\begin{aligned}\mathrm{ATSAF}(\boldsymbol{S}) &\ge K^2L^2V^2(2L-1)+(2L-1)\\ &\left(\sum_{\substack{j=2\\ j\equiv 0(\mathrm{mod}2)}}^{L}\left|\sum_{v=0}^{V-1}\xi_L^{-(j-1)v}\right|^2+\sum_{\substack{j=\frac{L}{2}+1\\ j\equiv 1(\mathrm{mod}2)}}^{L}\left|2\sum_{v=0}^{V-1}\xi_L^{-(j-1)v}\right|^2\right.\\ &\left.+\sum_{\substack{j=L+1\\ j\equiv 1(\mathrm{mod}2)}}^{2L-1}\left|\sum_{v=0}^{V-1}\xi_L^{(L-j)v}\right|^2+\sum_{\substack{j=L+1\\ j\equiv 0(\mathrm{mod}2)}}^{3L/2}\left|2\sum_{v=0}^{V-1}\xi_L^{(L-j)v}\right|^2\right)\\ &= K^2L^2V^2(2L-1)+(2L-1)\\ &\left(2\sum_{\substack{j=2\\ j\equiv 0(\mathrm{mod}2)}}^{L-2}\left|\sum_{v=0}^{V-1}\xi_L^{-jv}\right|^2+2\sum_{j=1}^{L-1}\left|\sum_{v=0}^{V-1}\xi_L^{-jv}\right|^2\right)\\ &= K^2L^2V^2(2L-1)+(2L-1)(3L-4V)V\end{aligned}\tag{32}$$

当$L/2<V\le L-1$时，将式(18)和式(29)代入式(10)，可得

$$\begin{aligned}\mathrm{ATSAF}(\boldsymbol{S}) &\ge K^2L^2V^2(2L-1)+(2L-1)\\ &\left(\sum_{\substack{j=2\\ j\equiv 0(\mathrm{mod}2)}}^{L}\left|\sum_{v=V-\frac{L}{2}}^{\frac{L}{2}-1}\xi_L^{-(j-1)v}\right|^2+\sum_{\substack{j=\frac{L}{2}+1\\ j\equiv 1(\mathrm{mod}2)}}^{L}\left|2\sum_{v=V-\frac{L}{2}}^{\frac{L}{2}-1}\xi_L^{-(j-1)v}\right|^2\right.\\ &\left.+\sum_{\substack{j=L+1\\ j\equiv 1(\mathrm{mod}2)}}^{2L-1}\left|\sum_{v=V-\frac{L}{2}}^{\frac{L}{2}-1}\xi_L^{(L-j)v}\right|^2+\sum_{\substack{j=L+1\\ j\equiv 0(\mathrm{mod}2)}}^{\frac{3L}{2}}\left|2\sum_{v=V-\frac{L}{2}}^{\frac{L}{2}-1}\xi_L^{(L-j)v}\right|^2\right)\\ &= K^2L^2V^2(2L-1)+(2L-1)\\ &\left(2\sum_{\substack{j=2\\ j\equiv 0(\mathrm{mod}2)}}^{L-2}\left|\sum_{v=V-\frac{L}{2}}^{\frac{L}{2}-1}\xi_L^{-jv}\right|^2+2\sum_{j=1}^{L-1}\left|\sum_{v=V-\frac{L}{2}}^{\frac{L}{2}-1}\xi_L^{-jv}\right|^2\right)\\ &= K^2L^2V^2(2L-1)+(2L-1)(L-V)(4V-L)\end{aligned}\tag{33}$$

当$V=L$时，容易得到，$\mathrm{ATSAF}(\boldsymbol{S})=K^2L^4(2L-1)$。因此，当$L/2<V\le L$时，有

$$\mathrm{ATSAF}(\boldsymbol{S})=K^2L^2V^2(2L-1)+(2L-1)(L-V)(4V-L)$$

当$\varphi(L,K,j)$满足(VI)和(VII)时，证明同理。

综上所述，定理1。　　证毕

4 三类达界的最优构造

对于含有K条长度为L的二元序列集$\boldsymbol{S}=[\boldsymbol{s}_1,\boldsymbol{s}_2,\cdots,\boldsymbol{s}_K]$，若其满足定理1中的ATSAF理论下界，则称其为ATSAF最优序列集。根据式(6)可知，ATSAF最优序列集$\boldsymbol{S}$等价于非周期时相循环扩展矩阵$\boldsymbol{S}_a$是TSC最优的。在本节中，将针对不同K和L的情况，设计几类ATSAF最优二元序列集。

定理2 设$\boldsymbol{S}=[\boldsymbol{s}_1,\boldsymbol{s}_2,\cdots,\boldsymbol{s}_K]$为含有$K$条长度为$L$的二元序列集。如果$\boldsymbol{S}$是ACS，则$\boldsymbol{S}$为ATSAF最优序列集。

证明 如果$\boldsymbol{S}$是ACS，则其满足式(8)。设多普勒移位$0\le v\le V-1$，$V\le L$。由于$\boldsymbol{s}_i^{(v)}=\boldsymbol{s}_i\odot\boldsymbol{f}^{(v)}$，其中，$\boldsymbol{f}^{(v)}=[\xi_L^{0\times v}\xi_L^{1\times v}\cdots\xi_L^{(L-1)\times v}]^{\mathrm{T}}$，则对于序列集$[\boldsymbol{s}_1^{(v)},\boldsymbol{s}_2^{(v)},\cdots,\boldsymbol{s}_K^{(v)}]$，有

$$\sum_{i=1}^{K}C_{\boldsymbol{s}_i^{(v)}}(\tau)=\sum_{i=1}^{K}C_{\boldsymbol{s}_i}(\tau)\xi_L^{-\tau v}=0,\tau\ne 0 \tag{34}$$

因此，对于$0\le v\le V-1$，$[\boldsymbol{s}_1^{(v)},\boldsymbol{s}_2^{(v)},\cdots,\boldsymbol{s}_K^{(v)}]$也为ACS，则有

$$\sum_{v=0}^{L-1}\sum_{k=1}^{K}\left(\boldsymbol{u}_k^{(0,v)}\right)^{\mathrm{T}}\left(\boldsymbol{u}_k^{(l,v)}\right)^{*}=\begin{cases}KLV, & l=0\\ 0, & l\ne 0\end{cases} \tag{35}$$

将式(35)代入式(11)可知，

$$\boldsymbol{d}_1\boldsymbol{d}_j^{\mathrm{H}}=\sum_{v=0}^{V-1}\sum_{k=1}^{K}\left(\boldsymbol{u}_k^{(0,v)}\right)^{\mathrm{T}}\left(\boldsymbol{u}_k^{(j-1,v)}\right)^{*}=0,2\le j\le L \tag{36}$$

因此，由引理1可知，$\mathrm{ATSAF}(\boldsymbol{S})=K^2L^2V^2(2L-1)$，则$\boldsymbol{S}$为ATSAF最优序列集。 证毕

定理3 设K和L为满足$K\equiv 2(\mathrm{mod}4)$和$L\equiv 1(\mathrm{mod}2)$的两个整数。设$N=K-2$，$\boldsymbol{H}_N$为N阶Hadamard矩阵。选择$\boldsymbol{H}_N$的前L行组成$\overline{\boldsymbol{H}}_{L\times N}$。定义

$$\boldsymbol{S}=[\overline{\boldsymbol{H}}_{L\times N},\boldsymbol{g}_1,\boldsymbol{g}_2] \tag{37}$$

其中，$\boldsymbol{g}_1=[\mathrm{g}_1(0),g_1(1),\ldots,g_1(L-1)]^{\mathrm{T}}$为ATSAF最优序列，换句话说，$\boldsymbol{g}_1$的$\varphi_1(L,1,j)$满足(20)；$\boldsymbol{g}_2$为对$\boldsymbol{g}_1$中的元素进行交替取反得到，即$g_2(n)=(-1)^n g_1(n)$，$0\le n\le L-1$，$\boldsymbol{g}_2$的$\varphi_2(L,1,j)$也满足(20)；则$\boldsymbol{S}$为ATSAF最优序列集。

证明 设$\boldsymbol{S}_1=\overline{\boldsymbol{H}}_{L\times N}$，$\boldsymbol{S}_2=[\boldsymbol{g}_1,\boldsymbol{g}_2]$。定义序列集$\boldsymbol{S}$，$\boldsymbol{S}_1$和$\boldsymbol{S}_2$的非周期时相循环扩展矩阵分别为$\boldsymbol{S}_a$，$\boldsymbol{S}_{1,a}$和$\boldsymbol{S}_{2,a}$。由于$\boldsymbol{g}_1$的$\varphi_1(L,1,j)$和$\boldsymbol{g}_2$的$\varphi_2(L,1,j)$均满足式(20)，则$\boldsymbol{S}_2=[\boldsymbol{g}_1,\boldsymbol{g}_2]$的$\varphi(L,2,j)$满足式(22)。设多普勒移位$0\le v\le V-1$，$1\le V\le L$，则有

$$\left|\boldsymbol{d}_{1,1}\boldsymbol{d}_{1,j}^{\mathrm{H}}\right|=\begin{cases}(K-2)LV, & j=1\\ 0, & 2\le j\le 2L-1\end{cases} \tag{38}$$

$$\left|\boldsymbol{d}_{2,1}\boldsymbol{d}_{2,j}^{\mathrm{H}}\right|=\begin{cases}2LV, & j=1\\ \varphi(L,2,j)\sum_{v=0}^{V-1}\xi_L^{-(j-1)v}, & 2\le j\le L\\ \varphi(L,2,j)\sum_{v=0}^{V-1}\xi_L^{(L-j)v}, & L+1\le j\le 2L-1\end{cases} \tag{39}$$

其中，$\boldsymbol{d}_j$，$\boldsymbol{d}_{1,j}$和$\boldsymbol{d}_{2,j}$分别表示$\boldsymbol{S}_a^{\mathrm{T}}$，$\boldsymbol{S}_{1,a}^{\mathrm{T}}$和$\boldsymbol{S}_{2,a}^{\mathrm{T}}$的第$j$行，$\varphi(L,2,j)$满足式(22)，即，要么满足(I)和(IV)，要么满足(II)和(III)。此外，由于$\left|\boldsymbol{d}_1\boldsymbol{d}_j^{\mathrm{H}}\right|=\left|\boldsymbol{d}_{1,1}\boldsymbol{d}_{1,j}^{\mathrm{H}}+\boldsymbol{d}_{2,1}\boldsymbol{d}_{2,j}^{\mathrm{H}}\right|$，根据引理1和定理1中情况3的证明可知，$\mathrm{ATSAF}(\boldsymbol{S})=K^2L^2V^2(2L-1)+4(2L-1)(L-V)V$，则$\boldsymbol{S}$为ATSAF最优序列集。 证毕

注2 文献[6]和文献[25]已证明长度为$L\in\mathcal{L}_{MF}=\{1,2,3,4,5,6,7,10,11,13,14,18\}$的单条最大MF (Merit Factor)序列可以达到ATSC理论界，称为ATSC最优序列。文献[14]的表II中列出了长度为$L\in\mathcal{L}_{MF}$的最大MF序列。由于最大MF序列为ATSC最优序列，则不同长度的最大MF序列的$\varphi_{\mathrm{MF}}(L,1,j)$分别满足式(20)，式(26)或式(29)。根据定理1的证明可知，最大MF序列也为ATSAF最优序列。

定理4 设K和L为满足$K\equiv 3(\mathrm{mod}4)$和$L\in\mathcal{L}_{MF}$的两个整数。设$\boldsymbol{H}_{K+1}$为$K+1$阶Hadamard矩阵，删除$\boldsymbol{H}_{K+1}$的全1列，并选择前L行构成矩阵$\overline{\boldsymbol{H}}_{L\times K}$。定义

$$\boldsymbol{S}=\mathrm{diag}\{\boldsymbol{g}\}\times\overline{\boldsymbol{H}}_{L\times K} \tag{40}$$

其中，$\boldsymbol{g}$是长度为L的序列。如果$\boldsymbol{g}$是ATSAF最优序列，则$\boldsymbol{S}$为ATSAF最优序列集。

证明 定理4的证明与文献[14]中附录D的证明相同，再结合定理1的证明，则

$$\mathrm{ATSAF}(\boldsymbol{S})=K^2L^2V^2(2L-1)+\begin{cases}(2L-1)(L-V)V, & L\equiv 1(\mathrm{mod}2)\\ (2L-1)LV, & L\equiv 0(\mathrm{mod}4),1\le V\le L/2\\ (2L-1)(L-V)L, & L\equiv 0(\mathrm{mod}4),L/2<V\le L\\ (2L-1)(3L-4V)V, & L\equiv 2(\mathrm{mod}4),1\le V\le L/2\\ (2L-1)(L-V)(4V-L), & L\equiv 2(\mathrm{mod}4),L/2<V\le L\end{cases} \tag{41}$$

证毕

下面通过一个例子来说明定理4。

例1　设$L=4$，$K=7$。删除8阶Hadamard矩阵的全1列，并选择前4行构成矩阵$\overline{\boldsymbol{H}}_{4\times 7}$，即

$$\overline{\boldsymbol{H}}_{4\times 7}=\begin{bmatrix} +++++++ \\ -+-+-+- \\ +--++-- \\ --++--+ \end{bmatrix} \tag{42}$$

其中，$+$和$-$ 分别表示1和-1。由文献[14]的表II可知，$\boldsymbol{g}=[++-+]^T$是长度为4的最大MF序列。根据定理4，构造大小为4×7的二元序列集

$$\boldsymbol{S}=\begin{bmatrix} +++++++ \\ -+-+-+- \\ -++--++ \\ --++--+ \end{bmatrix} \tag{43}$$

设$1\le V\le L=4$，$\boldsymbol{S}$的非周期时相循环扩展矩阵为$\boldsymbol{S}_a$，大小为$7\times 49V$。由定理1可知，当$1\le V\le L/2$时，$\boldsymbol{S}$的ATSAF理论下界为$5488V^2+28V$。当$L/2<V\le L$时，$\boldsymbol{S}$的ATSAF理论下界为$5488V^2+28(4-V)$。由于$\text{ATSAF}(\boldsymbol{S})=\text{TSC}(\boldsymbol{S}_a)$，则可以通过计算$\boldsymbol{S}_a$的TSC即可得到$\boldsymbol{S}$的ATSAF值。表1中列出了$\boldsymbol{S}$在不同多普勒移位$V$时的ATSAF值和理论下界。由于$\boldsymbol{S}$的ATSAF值和理论下界在不同多普勒移位$V$时相等，则定理4构造的$\boldsymbol{S}$为ATSAF最优序列集。

表 1　例1中序列集$\boldsymbol{S}$的ATSAF值和理论下界

V	1	2	3	4
ATSAF($\boldsymbol{S}$)	5516	22008	49420	87808
理论下界	5516	22008	49420	87808

5　结论

本文考虑到二元序列集$\boldsymbol{S}$在传输过程中可能产生的时移和多普勒移位，定义了非周期时相循环扩展矩阵$\boldsymbol{S}_a$。基于矩阵$\boldsymbol{S}_a$，给出了二元序列集的ATSAF理论下界，构造了不同参数下能够达到ATSAF理论下界的二元序列集。

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责任编辑： 廖海贝

Aperiodic Total Squared Ambiguity Function: Theoretical Bounds for Binary Sequence Sets and Optimal Constructions

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Abstract:

Objective In direct-sequence code division multiple access systems, the performance of spreading sequence sets is commonly evaluated using the total squared correlation metric. Traditional metrics such as total squared correlation and aperiodic total squared correlation are applicable only to synchronous communication systems and asynchronous systems with time shifts only, respectively. In modern high-speed mobile and satellite communications, the Doppler effect becomes significant. It causes both time and Doppler shifts in the received signal and leads to severe signal distortion. In communication scenarios that consider only time shift, the one-dimensional correlation function is typically used to measure system interference. However, in high-speed mobile environments the Doppler effect appears during signal transmission. Both time shift and Doppler shift of the sequence must therefore be considered simultaneously. In such cases, the two-dimensional ambiguity function should replace the one-dimensional correlation function. To mitigate Doppler effects, recent studies have focused on the design of Doppler-resilient sequences for mobile channels. Existing work mainly studies theoretical bounds of the ambiguity function, particularly the maximum ambiguity magnitude. Sequence sets are then constructed to achieve or asymptotically approach these bounds. This study instead examines the overall ambiguity function performance of binary sequence sets in asynchronous communication, namely the Aperiodic Total Squared Ambiguity Function (ATSAF). The objectives are as follows. First, the theoretical lower bound for the ATSAF of binary sequence sets is derived. Second, several classes of optimal binary sequence sets that achieve this bound are constructed based on the derived ATSAF bound.

Methods The aperiodic time-phase cycling extension matrix $\boldsymbol{S}_a$ is defined for a binary sequence set $\boldsymbol{S}$ consisting of K sequences of length L to account for both time shifts and Doppler shifts. This definition converts the computation of the ATSAF for the sequence set $\boldsymbol{S}$ into the calculation of the total squared correlation of the matrix $\boldsymbol{S}_a$. The theoretical lower bounds for the ATSAF of the binary sequence set $\boldsymbol{S}$ are then derived for different combinations of the set size K, sequence length L, and Doppler shift V. To design binary sequence sets that achieve these ATSAF lower bounds, it is first proven that binary aperiodic complementary sets form ATSAF-optimal binary sequence sets. Furthermore, two additional classes of optimal binary sequence sets are constructed using Hadamard matrices and specific sequences. These sets are proven to achieve the theoretical ATSAF lower bound.

Results and Discussions Existing studies mainly examine the maximum ambiguity magnitude of sequence sets, whereas this study analyzes the overall ambiguity function performance. The one-dimensional aperiodic total squared correlation analysis for asynchronous communication with delay only, studied by Ganapathy et al., is extended to the two-dimensional ATSAF, which considers both time delay and Doppler shift. First, the aperiodic time-phase cycling extension matrix $\boldsymbol{S}_a$ is defined for a binary sequence set $\boldsymbol{S}$ (Definition 3). The theoretical lower bounds for the ATSAF of the binary sequence set $\boldsymbol{S}$ are then derived for different parameters, including set size K, sequence length L, and Doppler shift V (Theorem 1). When the Doppler shift $V = 1$, the derived ATSAF bound reduces to the aperiodic total squared correlation bound. Binary sequence sets that achieve these ATSAF bounds maintain the overall cross-interference energy in the two-dimensional delay-Doppler domain at its theoretical minimum. To construct such sequence sets, it is first proven that binary aperiodic complementary sets are ATSAF-optimal binary sequence sets (Theorem 2). Furthermore, two further classes of ATSAF-optimal binary sequence sets are constructed using Hadamard matrices and specific sequences (Theorems 3 and 4). Finally, an example demonstrates that the sequence set constructed in Theorem 4 is ATSAF-optimal (Example 1).

Conclusions In high-speed mobile communication scenarios, Doppler effects cause distortion in received signals. By defining the aperiodic time-phase cycling extension matrix $\boldsymbol{S}_a$ for a binary sequence set $\boldsymbol{S}$, the theoretical lower bound for the ATSAF is derived. This bound specifies the minimum theoretical value of the total energy of the binary sequence set S in the two-dimensional delay-Doppler domain. When Doppler shifts are not considered, the derived ATSAF bound reduces to the aperiodic total squared correlation bound. Furthermore, three classes of ATSAF-optimal binary sequence sets that achieve this theoretical bound are constructed using binary aperiodic complementary sets, Hadamard matrices, and specific sequences. These sequence sets maintain the overall cross-interference energy at the theoretical minimum in the two-dimensional delay-Doppler domain.

Key words: Aperiodic total squared ambiguity function; Ambiguity function; Aperiodic total squared correlation